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Ext for Blocks with Cyclic Defect Groups

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1. INTRODUCTION

Throughout this paper, k denotes an algebraically closed field. We study the Ext-groups for a block with a cyclic defect group, or more generally, for a k-algebra given by a Brauer tree, which was introduced by Reiten [8] (cf. Brauer [2]).

THEOREM 1.1. *Let A be a k-algebra given by a Brauer tree with e edges and with the multiplicity m of the exceptional vertex. If i is an odd integer and j is an even integer, then there are m nonprojective indecomposable A-modules M for which $\mathrm{Ext}_A^n(S, M) \neq 0$ only for $n \equiv i \pmod{2e}$ and $n \equiv j \pmod{2e}$, where S is a fixed simple module whose projective cover is uniserial.*

Here *uniserial* means *with a unique composition series*. In fact we choose these modules M from the indecomposable modules classified by Janusz [7] and use the canonical walk by Alperin and Janusz [1] to state this precise result (Theorem 3.1). An outline for the computation of $\mathrm{Ext}_A^n(S, M)$ is as follows: For a nonprojective indecomposable module M, we examine the property of $\Omega^{-1}M$ (Proposition 3.2) and then use $\mathrm{Ext}_A^n(S, M) \simeq \mathrm{Ext}_A^1(S, \Omega^{1-n}M)$, where Ω is Heller's loop-space operation [6]. We know that $\mathrm{Ext}_A^1(S, M) \neq 0$ if and only if there is the exact sequence $0 \to M \to W \to S \to 0$ with W indecomposable (see the proof of Proposition 3.3).

We remark that, if A is the principal block of kG and S is the trivial kG-module, then $\mathrm{Ext}_A^n(S, M)$ is the cohomology group $H^n(G, M)$, where G is a finite group with a cyclic Sylow p-subgroup and k has a characteristic p. Finally we refer to [4, Chap. VII; and 3] for blocks with cyclic defect groups and for the functor Ext, respectively.

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2. PRELIMINARIES

Here we state basic results. All vector spaces over k are finite-dimensional. A tree means a nontrivial plane graph which is connected and has no cycle. According to Reiten [8] we consider a k-algebra given by a Brauer tree as a generalization of a block with a cyclic defect group.

(2.1) *The canonical walk* (Alperin and Janusz [1]). Let A be a k-algebra given by a Brauer tree: The e edges correspond to the simple modules and their projective covers. Every vertex P has the multiplicity $m(P)$. There is a unique vertex called exceptional. If P is exceptional, $m(P) = m \geqslant 1$. Otherwise $m(P) = 1$. Around each vertex, there is a counter-clockwise ordering of the edges. Let $E = P_1 P_2$ be the edge corresponding to a projective indecomposable module U; j denotes 1 and 2. If the sequence of edges $E_j, \ldots, E$ represents this ordering around P_j, then we have $JU = U_1 + U_2$, where U_j is uniserial and its composition factors from the top down correspond to $m(P_j)$ times repetition of $E_j, \ldots, E$. Here J is the radical of A.

Now we let S be a fixed simple module whose projective cover is uniserial. Then S corresponds to an edge $E = PQ$, where $m(P) = 1$ and E is a unique edge incident with P [7, Corollary 7.3]. Put $Q_0 = P$, $F_0 = E$, and $Q_1 = Q$. For $i \geqslant 0$ let F_{i+1} be the edge immediately following F_i around Q_{i+1} and let Q_{i+2} be the vertex such that $F_{i+1} = Q_{i+1} Q_{i+2}$. Thus we get the canonical walk $Q_0, F_0, Q_1, F_1, Q_2, \ldots, Q_i, F_i, Q_{i+1}, \ldots$. Each edge appears twice in the walk $Q_0, F_0, Q_1, \ldots, Q_{2e-1}, F_{2e-1}, Q_{2e}$.

(2.2) *Nonprojective indecomposable modules* (Janusz [7], see also Reiten [8]). Given any edges E and F we let $P_0, E_1, P_1, \ldots, P_{s-1}, E_s, P_s$ be the path which begins at E and ends at F ($E_1 = E$ and $E_s = F$), or the shortest walk that begins at E, goes through the exceptional vertex and ends at F (if it is not a path). Let $M_1, \ldots, M_t$ be the sequence of modules with the following properties: For $1 \leqslant i \leqslant t$, M_i has a unique maximal submodule $JM_i = M_{i,1} \oplus M_{i,2}$ and $M_{i,j}$ is uniserial. For $1 \leqslant i \leqslant t-1$ there is an isomorphism $\psi_i \colon \operatorname{soc} M_{i,2} \to \operatorname{soc} M_{i+1,1}$; soc $M_{1,1}$ (if $M_{1,1} \neq 0$), M_1/JM_1, soc $M_{1,2}$, M_2/JM_2, soc $M_{2,2}, \ldots,$ soc $M_{t-1,2}$, M_t/JM_t, soc $M_{t,2}$ (if $M_{t,2} \neq 0$) correspond to $E_1, \ldots, E_s$. Put $M = (M_1, \ldots, M_t) = M_1 \oplus \cdots \oplus M_t/X$, where X is the submodule of $M_1 \oplus \cdots \oplus M_t$ whose elements are $\{(x_1, \psi_1(x_1) + x_2, \ldots, \psi_{t-2}(x_{t-2}) + x_{t-1}, \psi_{t-1}(x_{t-1})) \mid x_i \in \operatorname{soc} M_{i,2}\}$. Every nonprojective indecomposable module is isomorphic to a unique $(M_1, \ldots, M_t)$ up to the reversal of the order of $M_1, \ldots, M_t$.

3. Ext-Groups

For integers i and j we consider the condition (C) $0 < i < j < i + 2e$; $i - j$ is odd; $F_i = F_j$ or F_i is not on the path combining F_j and the exceptional vertex.

THEOREM 3.1. *Let M be a nonprojective indecomposable A-module, where A is a k-algebra given by a Brauer tree with e edges. Let S be a fixed simple module whose projective cover is uniserial; i and j denote the integers satisfying the condition (C).*

We have $\mathrm{Ext}^n_A(S, M) \neq 0$ only for $n \equiv i \pmod{2e}$ and $n \equiv j \pmod{2e}$ if and only if M is isomorphic to the A-module $(M_1, ..., M_t)$ with the property (P): F_i corresponds to $\mathrm{soc}\, M_{1,1}$ if the edge denoted by F_i does not appear between Q_{i+1} and Q_j in the canonical walk. Otherwise F_i corresponds to M_1/JM_1 $(M_{1,1} = 0)$; F_j corresponds to $\mathrm{soc}\, M_{t,2}$ or M_t/JM_t $(M_{t,2} = 0)$.

If $M = (M_1, ..., M_t)$ has the property (P), $\mathrm{Ext}^i_A(S, M) \simeq \mathrm{Ext}^j_A(S, M) \simeq k$.

To compute $\mathrm{Ext}^n_A(S, M)$ we use Fig. 1. Walking around the Brauer tree we keep to the right side of an edge and write F_i there. Then Fig. 1 represents the path which begins at F_i and ends at F_j, where we omit the vertices.

If we define $M = (M_1, ..., M_t)$ for this path, the property (P) is restated as follows: If F_i is written under [over] the left edge, F_i corresponds to $\mathrm{soc}\, M_{1,1}$ [M_1/JM_1]; If F_j is written under [over] the right edge, F_j corresponds to M_t/JM_t [$\mathrm{soc}\, M_{t,2}$]. For the shortest walk that begins at F_i, goes through the exceptional vertex, and ends at F_j, property (P) is restated as above except when $F_i \neq F_j$ and F_j is on the path combining F_i and the exceptional vertex. Then the property (P) is restated as follows: soc $M_{1,1}$ or M_1/JM_1 is as above; If F_j is written under [over] the right edge, F_j corresponds to $\mathrm{soc}\, M_{t,2}$ [M_t/JM_t]. Here brackets mean *respectively*.

We recall Heller's loop-space operation Ω [6] (see also Green [5]): Let M be a nonprojective indecomposable module. If X is the projective cover of M, define ΩM to be the kernel of the essential epimorphism $X \to M$. If Y is the injective hull of M, put $\Omega^* M = Y/M$. Then Ω and Ω^* are inverse permutations of the set of isomorphism classes of nonprojective indecomposable modules [6, Proposition 1].

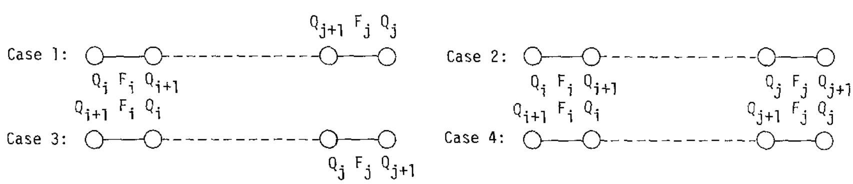

FIG. 1. The path which begins at F_i and ends at F_j; i and j are integers such that $0 < i < j < i + 2e$ and $i - j$ is odd.

PROPOSITION 3.2. *For a nonprojective indecomposable module $M=(M_1,\dots,M_t)$ with the property (P) for i and j satisfying the condition (C), Ω^*M is isomorphic to a module $L=(L_1,\dots,L_r)$ with the property (P) for $i-1$ and $j-1$ [for $j-1$ and $i+2e-1$] if $i-1$ and $j-1$ satisfy the condition (C) [otherwise].*

Proof. If $i-1$ and $j-1$ do not satisfy the condition (C), then $j-1$ and $i+2e-1$, and j and $i+2e$ satisfy the condition (C). Thus we may assume that $i-1$ and $j-1$ satisfy the condition (C).

We treat only the case where the edge denoted by F_i does not appear between Q_{i+1} and Q_j in the canonical walk, but the edge denoted by F_j does, and where $M=(M_1,\dots,M_t)$ is defined for the path which begins at F_i and ends at F_j (Case 1 of Fig. 1).

The injective hull of M is isomorphic to $U_1\oplus\cdots\oplus U_{t+1}$, where $U_1,\dots,U_{t+1}$ are the injective hull of soc $M_{1,1}$, soc $M_{1,2}$, soc $M_{2,2},\dots$, soc $M_{t-1,2}$, soc $M_{t,2}$. Then Ω^*M is isomorphic to the $L=(L_1,\dots,L_{t+1})$ with $L_1/JL_1\simeq \operatorname{soc} M_{1,1}$ and $L_{t+1}/JL_{t+1}\simeq \operatorname{soc} M_{t,2}$. We have $\sum_{i=1}^{t+1}\lg(U_i)=\sum_{i=1}^{t+1}\{\lg(U_{i,1})+\lg(U_{i,2})\}$, $\lg(M)=1+\sum_{i=1}^{t}\{\lg(M_{i,1})+\lg(M_{i,2})\}$, and $\lg(L)=1+\sum_{i=1}^{t+1}\{\lg(L_{i,1})+\lg(L_{i,2})\}$. Here $\lg(\cdot)$ denotes the composition length of a module. Also we have $1+\lg(L_{1,1})\leqslant\lg(U_{1,1})$, $1+\lg(L_{t+1,2})\leqslant\lg(U_{t+1,2})$, $\lg(M_{i,2})+\lg(L_{i+1,1})\leqslant\lg(U_{i+1,1})$, and $\lg(L_{i,2})+\lg(M_{i,1})\leqslant\lg(U_{i,2})$ for $1\leqslant i\leqslant t$. Since $\sum_{i=1}^{t+1}\lg(U_i)=\lg(M)+\lg(L)$, all the inequalities are the equalities. This proves the proposition.

(If $i=1$, we consider the property (P) for 0 and $j-1$ or for $j-1$ and $2e$.)

PROPOSITION 3.3. *For a indecomposable module M we have $\operatorname{Ext}_A^1(S,M)\neq 0$ if and only if M is isomorphic to the $(M_1,\dots,M_t)$ with one of the properties: $P_0=Q_1$, $E_1=F_1$, $P_1=Q_2$, and $M_{1,1}\neq 0$; $P_0=Q_2$, $E_1=F_1$, $P_1=Q_1$, and $M_{1,1}=0$. If $\operatorname{Ext}_A^1(S,M)\neq 0$, then $\operatorname{Ext}_A^1(S,M)\simeq k$.*

Proof. If $\operatorname{Ext}_A^1(S,M)\neq 0$ for $M=(M_1,\dots,M_t)$, the exact sequence $\operatorname{Hom}_A(U,M)\to\operatorname{Hom}_A(JU,M)\to\operatorname{Ext}_A^1(S,M)\to 0$ implies that there exists $f\in\operatorname{Hom}_A(JU,M)$ not equal to any $g\in\operatorname{Hom}_A(U,M)$ on JU, where U is the projective cover of S. Suppose that $f(JU)$ is nonsimple. [8, Lemma 1.1] shows that $f(JU)$ is contained in a unique M_i. Here M_i is regarded as a submodule of M in a natural way. If $f(JU)\neq M_i$, there exists $g\in\operatorname{Hom}_A(U,M)$ equal to f on JU. Thus $f(JU)=M_i$ for $i=1$ or $i=t$. We may assume that $i=1$. Then we have $P_0=Q_2$, $E_1=F_1$, $P_1=Q_1$, and $M_{1,1}=0$. Next suppose that $f(JU)$ is simple. Then $f(JU)=\operatorname{soc} M_{1,1}$ or $f(JU)=\operatorname{soc} M_{t,2}$ for the same reason. If we assume $i=1$, we have $P_0=Q_1$, $E_1=F_1$, $P_1=Q_2$, and $M_{1,1}\neq 0$.

Conversely let $M=(M_1,\dots,M_t)$ with $P_0=Q_1$, $E_1=F_1$, $P_1=Q_2$, and $M_{1,1}\neq 0$. We have the nonsplit exact sequence $0\to M\to W\to S\to 0$, where $W=(U/J^2U,\ M_1,\dots,M_t)$. Suppose that $M=(M_1,\dots,M_t)$ has the property

$P_0 = Q_2$, $E_1 = F_1$, $P_1 = Q_1$, and $M_{1,1} = 0$. Then put $W = (U_1, M_2, \ldots, M_t)$, where U_1 is the homomorphic image of U such that $\operatorname{soc} U_1 \simeq \operatorname{soc} M_{1,2}$.

Finally this argument shows that there is the exact sequence $0 \to M \to W \to S \to 0$ with $\operatorname{Ext}_A^1(S, W) = 0$, if $\operatorname{Ext}_A^1(S, M) \neq 0$. Hence the exact sequence $\cdots \to \operatorname{Hom}_A(S, S) \to \operatorname{Ext}_A^1(S, M) \to \operatorname{Ext}_A^1(S, W) \to \cdots$ implies $\operatorname{Ext}_A^1(S, M) \simeq k$.

We restate Proposition 3.3 in the suitable form for the proof of Theorem 3.1:

COROLLARY 3.4. *For an indecomposable module* $M = (M_1, \ldots, M_t)$ *with the property* (P) *for integers* i *and* j *such that* $0 < i < j < i + 2e$ *and* $i - j$ *is odd, we have* $\operatorname{Ext}_A^1(S, M) \neq 0$ *if and only if* $i \equiv 1 \pmod{2e}$ *or* $j \equiv 1 \pmod{2e}$.

Proof of Theorem 3.1. We prove the if part, which deduces the only-if part, for every nonprojective indecomposable module is isomorphic to a module $(M_1, \ldots, M_t)$ with the property (P) for some integers i and j such that $0 < i < j < i + 2e$ and $i - j$ is odd. Suppose that $M = (M_1 \ldots, M_t)$ has the property (P) for fixed integers i and j such that $0 < i < j < i + 2e$ and $i - j$ is odd. If $n \equiv i \pmod{2e}$, Proposition 3.2 implies that $\Omega^{1-n}M$ is isomorphic to the $(L_1, \ldots, L_r)$ with the property (P) for 1 and $j - i + 1$ or for $j - i + 1$ and $1 + 2e$. By Corollary 3.4 we have $\operatorname{Ext}_A^1(S, \Omega^{1-n}M) \neq 0$. This is isomorphic to $\operatorname{Ext}_A^n(S, M)$, and so $\operatorname{Ext}_A^n(S, M) \neq 0$. In the same way we have $\operatorname{Ext}_A^n(S, M) \neq 0$ for $n \equiv j \pmod{2e}$. If $n \not\equiv i \pmod{2e}$ and $n \not\equiv j \pmod{2e}$, $i - (n-1) \not\equiv 1 \pmod{2e}$ and $j - (n-1) \not\equiv 1 \pmod{2e}$. Thus Corollary 3.4 shows $\operatorname{Ext}_A^n(S, M) = 0$. The final assertion follows from Proposition 3.3.

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